Theoretical Models of the Structure of Protoplanetary Disks Les Houches 2013

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1 Introduction

Planets are by-products of star formation. When a star forms from a rotating molecular cloud core, the angular momentum conservation law forces this collapsing cloud matter to form a rotating accretion disk around the newly forming star. As a result of "turbulent viscosity" much of this matter is spiralling inward and lands on the star. We therefore call this disk a *protostellar disk*. After a few 10^5 years, however, most of the matter of the disk has ended up on the star. The remaining disk is still an active accretion disk, but with a mass $M_{\rm disk} \ll M_*$. Over the next 10 million years or so this disk will gradually empty itself onto the star. However, the fact that this disk stays there for $\sim 10^7$ years means that there is enough time for planets to form out of the disk. This is why we call the disk, in this late stage, a *protoplanetary disk*.

The idea that the solar system was born out of a protoplanetary disk originates back to Kant and Laplace, who realized that the fact that all planets in the solar system rotate around the sun in the same plane and in the same direction must mean that they must all have a common origin in an flattened rotating "Urnebel" or "solar nebula". By looking at the mass distribution of planets in the *current* solar system one can calculate the minimal amount of matter that must have been present in this disk 4.56 billion years ago. This leads to the well-known "Minimum Mass Solar Nebula" (MMSN) model (Weidenschilling 1977; Hayashi 1981; Thommes & Duncan 2006), which can be written as:

$$\Sigma_{\rm gas}^{\rm MMSN}(r) = 1700 \left(\frac{r}{1\,{\rm AU}}\right)^{-3/2} {\rm g\,cm^{-2}} \quad (1)$$

$$\Sigma_{\rm dust}^{\rm MMSN}(r) = 7 \left(\frac{r}{1\,{\rm AU}}\right)^{-3/2} {\rm g\,cm^{-2}} \quad (2)$$

$$\Sigma_{\rm ice}^{\rm MMSN}(r) = 22 \left(\frac{r}{1\,{\rm AU}}\right)^{-3/2} {\rm g\,cm^{-2}} \qquad \text{for } r > 2.7\,{\rm AU} \quad (3)$$

where the ice is assumed to be present only for $r > 2.7\,{\rm AU}$ (we will, however, look more closely at this "snow line" later). A surface density Σ is defined as the vertical integral of the density:

$$\Sigma(r) \equiv \int_{-\infty}^{+\infty} \rho(r,z)dz \quad (4)$$

Since the mid-90s protoplanetary disks are routinely observed around young stars (T Tauri stars and Herbig Ae/Be stars), so we know now much better what such disks look like. It is found that protoplanetary disks are for most part relatively cool ($T \sim 10 \cdots 10^3$K), so that dust can survive. Indeed, these disks are found to be very dust-rich, and their radiative properties are dominated by the dust continuum opacity.

Unfortunately current observational facilities are not yet powerful enough to really observe deep into these disks into the regions where planets actually form. This has two reasons:

1. The disks are optically thick at most wavelengths, meaning that we can usually only see the surface layers. Only the outer disk regions ($r \gtrsim 40$ AU) become optically thin, but only at millimeter wavelengths (at optical and infrared wavelengths they remain optically thick).
2. Most young stars with disks are at distances of $\gtrsim$ 100 pc, meaning that it requires extreme angular resolution to resolve the planet-forming regions.

This means that even today, with the many observations we have of protoplanetary disks, we still do not really know with certainty what the structure of these disks is! In particular, we still have no real answers to the following questions:

- **Dust-to-gas ratio d/g:** It is very hard to measure the amount of gas in disks, and we also do not really know the opacities of the dust very well. This means that we have very little knowledge of the dust-to-gas ratio (how much gram of dust is there in the disk for every gram of gas). Modelers typically *guess* that $d/g = 0.01$, which is roughly the value that is valid for the interstellar medium. But this is nothing more than an educated guess.

- **Disk mass $M_{\rm disk}$:** When astronomers talk about an observed disk mass, they mean "I have measured the dust continuum emission at millimeter wavelengths, assumed that the disk is optically thin, assumed that I known the opacity of the dust, and assumed that the $d/g = 0.01$". If anything, they have measured the dust mass of the disk. The conversion into disk mass (i.e. gas mass, since gas dominates by assumption over the dust) is based on this very shaky $d/g = 0.01$ assumption.

- **Surface density profile $\Sigma(r)$:** Millimeter observations probe the outer disk regions ($r \gtrsim 40$ AU). They say little about the planet-forming regions ($r \lesssim 40$ AU). The surface density of matter in these inner regions is inferred by extrapolation, often simply assuming a powerlaw $\Sigma(r) \propto r^{-3/2}$ or so. This powerlaw is, however, an assumption, not a fact.

- **The midplane temperature $T(r, z = 0)$:** While infrared observations can tell us with relatively high accuracy what the temperature of the disk is at the photosphere of the disk (henceforth "disk surface"), it does not give any information about the temperature in the disk midplane, where the planets form.

Since the structure of protoplanetary disks stands at the very basis of all planet formation theories, these uncertainties pose a real problem. For now the best we can do is make plausible theoretical models of these disks and make sure that their observable properties are at least consistent with what we observe. In this lecture we will develop such a plausible model.

2 A steady-state viscous accretion disk model

Let us make a model of a protoplanetary disk as an axisymmetric geometrically thin accretion disk. In this section we will not concern ourselves with the disk's vertical structure yet (we will do that later). Therefore we will describe the matter distribution with the surface density $\Sigma(r)$ and the temperature with the midplane temperature $T(r)$.

If the gas in the disk is perfectly non-viscous, and rotates in a keplerian fashion around the star, then each fluid element has a specific angular momentum $l = \Omega_K r^2$, where Ω_K is the keplerian angular frequency:

$$\Omega_K = \sqrt{\frac{GM_*}{r^3}} \tag{5}$$

where G is the gravitational constant and M_* is the stellar mass. We thus have $l \propto \sqrt{r}$. If this parcel wants to slowly spiral inward to the star (i.e. accretion), then it must evidently lower its specific angular momentum l, because r decreases. Without any torque on this fluid element, it will thus not accrete and instead rotate forever in a circular orbit around the star. Viscosity can, however, lead to an *outward redistribution* of angular momentum: the inner disk transports angular momentum to the outer disk, so that the inner disk moves inward, while the outer disk moves outward.

It turns out that the molecular viscosity $\nu_{\rm mol}$ in astrophysical accretion disks is extraordinarily low. So low, that the accretion would take billions of years or more. However, there are a number of theoretical arguments to presume that these disks are turbulent, and that turbulence can lead to an *effective turbulent viscosity*. We will not go into the details of this turbulent viscosity, other than noting that if you ever hear the term "magnetorotational instability (MRI)" or "baroclinic instability (BI)" or "dead zones", then you are hearing astronomers talk about the processes that produce turbulent viscosity in disks. Instead, we are going to use the standard

"α-viscosity prescription":

$$\nu = \alpha \frac{c_s^2}{\Omega_K} \tag{6}$$

where

$$c_s = \sqrt{\frac{kT}{\mu m_p}} \tag{7}$$

is the *isothermal sound speed*, where k is the Boltzmann constant, mu is the mean molecular weight (for our purposes $\mu = 2.3$) and m_p is the proton mass. The parameter α is a dimensionless number that can be regarded as a turbulent strength parameter. For physical reasons it must be $\alpha \lesssim 1$ and it is typically assumed to be $\alpha \simeq 10^{-3} \cdots 10^{-2}$. This assumption is based on models of MRI (see above) and on observations of protoplanetary disks (Hartmann et al. 1998). However, in the end the value of α is again just an educated guess, and it is sometimes scathingly called an "ignorance parameter". The form of Eq. (6) may look completely arbitrary at present, but there is some idea behind it, which has to do with the size and speed of turbulent eddies. However, let's postpone this analysis to Section A.

Now let us assume that we have a steady-state accretion disk. In that case it turns out (without proof) that the inward velocity of the gas is:

$$v_r = -\frac{3}{2}\frac{\nu}{r} \tag{8}$$

Let us put in some typical numbers: $r = 1\,$AU, $T = 300\,$K, $M_* = M_\odot$ and $\alpha = 0.01$ we obtain $v_r = -54\,$cm/s. As you can see, this is a fairly small velocity, considering that the azimuthal velocity is the Kepler velocity, which is $v_K = \Omega_K r = 2.98 \times 10^6$ cm/s = 29.8 km/s. It is also much smaller than the sound speed, which is $c_s = 1.04 \times 10^5$ cm/s.

With the radial inward velocity we can define the *accretion rate* $\dot{M}$ as:

$$\dot{M} = -2\pi r \Sigma v_r \tag{9}$$

For a non-steady-state disk this could be a function of r, but if we have a steady-state, then $\dot{M}$ must be constant with r, because otherwise we would have a time-dependent pile-up or depletion of mass somewhere (m mass conservation law). If, in fact, we insert again Eq. (8) then we obtain the final expression for the accretion rate in steady-state disks:

$$\dot{M} = -2\pi r \Sigma v_r \equiv 3\pi\Sigma\nu = \text{constant} \tag{10}$$

Let us put in the typical numbers again, and take $\Sigma = 10^3$ g/cm^2. We then get $\dot{M} = 5.1 \times 10^{18}$ g/s = $8 \times 10^{-8}\,M_\odot$/year, which is indeed a typical value for measured accretion rates of protoplanetary disks.

Let us now *assume* that the surface density $\Sigma(r)$ and the midplane temperature $T(r)$ are powerlaw functions of r:

$$\Sigma(r) \propto r^p \qquad \text{and} \qquad T(r) \propto r^q \tag{11}$$

We then have

$$\nu \propto \frac{T}{\Omega_K} \propto r^{q+3/2} \tag{12}$$

This then implies

$$v_r \propto r^{q+1/2} \tag{13}$$

Putting this into Eq. (10) gives

$$r^{p+q+3/2} \propto \text{constant} \tag{14}$$

and thus:

$$p + q = -\frac{3}{2} \tag{15}$$

Let us apply this to the MMSN, which has $p = -3/2$: it would imply $q = 0$, meaning that the midplane temperature would be constant with radius! This clearly shows that the MMSN is not consistent with standard viscous accretion disk theory. Either viscous accretion disk theory is wrong (or incomplete), which is very well possible given the number of assumptions we made, or the MMSN is incorrect.

In Section 5 we will see that a more realistic powerlaw dependence of the temperature would be $q = -1/2$ or $q = -3/4$ or even steeper, leading to $p = -1$ or $p = -3/4$ or even shallower.

One possible explanation for disks that do not obey $p + q = -3/2$ is that α may not be constant throughout the disk. In particular the above mentioned "dead zones", where α can drop to very low values ($\alpha \sim 10^{-6}$ or so), might explain some of this.

3 Vertical density structure of protoplanetary disks

Now that we have a bit a feeling for how the matter is distributed radially in the disk (Section 2), let us now look at the vertical distribution of matter. Like with the radial structure, the vertical density structure is related to the vertical temperature structure. So let us, for now, make the simplification that the temperature $T(r, z)$ is independent of z, i.e. that $\partial T/\partial z = 0$, and thus that $\partial c_s^2/\partial z = 0$. We will see in Section 5 that this is not correct, but it is not so bad either, so for now it will do.

To very good approximation the vertical hydrostatic balance equation in the disk is:

$$\frac{\partial p}{\partial z} = -\rho \frac{GM_*}{r^3} z \equiv -\rho \Omega_K^2 z \tag{16}$$

With $p = \rho c_s^2$ and with $\partial c_s^2/\partial z = 0$ we can turn this into

$$\frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{\Omega_K^2}{c_s^2} z \tag{17}$$

The solution to this differential equation is:

$$\rho(z) = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right) \tag{18}$$

where we defined the *pressure scale height* H as

$$H = \frac{c_s}{\Omega_K} \tag{19}$$

With the definition of the surface density Σ (Eq.4) we can write ρ_0 in terms of Σ and Eq. (18) can be written as

$$\rho(z) = \frac{\Sigma}{H\sqrt{2\pi}} \exp\left(-\frac{z^2}{2H^2}\right) \tag{20}$$

It should be kept in mind that H is a function of r. If we have the typical temperature profile for an irradiated disks of $T \propto r^{-1/2}$ (see Section 5) then $c_s \propto r^{-1/4}$ so that

$$H \propto r^{-1/4+6/4} \propto r^{5/4} \tag{21}$$

In other words: the ratio $H/r \propto r^{1/4}$, i.e. it increases with r. The disk thus has a *flaring geometric shape*, which is important for what we will discuss in Section 5.

Let us put in some typical numbers: $r = 1\,$AU, $T = 300\,$K, $M_* = M_\odot$ we find $H/r = 0.035$, i.e. the disk is geometrically thin, as expected.

4 Heating and cooling processes in protoplanetary disks

The main missing ingredient in our model so far is any knowledge of the temperature structure. To make a model of the thermal structure of the disk we must study the heating and cooling processes in the disk.

4.1 Viscous heating due to the accretion process

We saw that accretion can only happen if we have viscosity. But we also know from experience that viscosity (friction) leads to the production of heat. The heating per gram of gas is proportional to the viscosity coefficient ν and the square of the shear. In an accretion disk this amounts to $q_+ = \nu(rd\Omega_K/dr)^2$. The total heating per cm^2 is then

$$Q_+^{\rm accr} = \Sigma\nu\left(r\frac{d\Omega_K}{dr}\right)^2 = \frac{9}{4}\Sigma\nu\Omega_K^2 \tag{22}$$

If we insert $\dot{M} = 3\pi\Sigma\nu$ then we get

$$Q_+^{\rm accr} = \frac{3}{4\pi}\dot{M}\Omega_K^2 \tag{23}$$

This makes sense, since it shows that the release of heat is proportional to the amount of matter that accretes. Note that with $\dot{M}$ constant, $Q_+^{\rm accr}(r)$ goes as

$$Q_+^{\rm accr}(r) \propto r^{-3/2} \tag{24}$$

Suppose now that the viscous heating would be the only heating process in the disk. Somehow the disk must get rid of this heat, otherwise it would become hotter and hotter and eventually reaches the virial temperature and accretion would stop. To extremely good approximation one can say that the disk must radiate the entire Q_+ away again. If we assume that the disk's two surfaces can radiate as Planck functions with temperature $T_{\rm eff}$ then the cooling rate is

$$Q_- = 2\sigma_{\rm SB}T_{\rm eff}^4 \tag{25}$$

with $\sigma_{\rm SB}$ the Stefan-Boltzmann constant. The factor of two is due to the two sides of the disk.

Equating $Q_- = Q_+$ then gives

$$T_{\rm eff} = \left\{\frac{3}{8\pi\sigma_{\rm SB}}\dot{M}\Omega_K^2\right\}^{1/4} \tag{26}$$

We see that if accretion is the only heating process, then the effective temperature goes as

$$T_{\rm eff} \propto \dot{M}^{1/4}\,r^{-3/4} \tag{27}$$

for a steady-state disk. This is a very robust result and depends much less on various uncertain assumptions. It is simply a reflection of the fact that the gravitational energy that is set free due to accretion must be radiated away. It should be kept in mind, though, that Eqs.(23, 26) require an additional correction factor close to the inner edge of the disk. This is, however, rarely of concern for those of us who are interested in protoplanetary disks.

4.2 Heating through irradiation by the star

The other main heating process in protoplanetary disks is irradiation. In the later stages of protoplanetary disk evolution it is in fact the dominant heating process.

Protoplanetary disks are optically thick. That means that the radiation from the star will not be able to pass through these disks. Instead, the radiation will be absorbed in the surface layers of the disk. However, *where* this radiation is absorbed depends entirely on the geometric shape of the disk. We can distinguish between two main shapes: *flaring disks* and *self-shadowed disks*. The difference lies in the height above the midplane H_s where the disk becomes transparent, or in other words, the height of the photosphere of the disk. Let us call H_s the *surface height*. We distinguish:

$$\frac{d}{dr}\left(\frac{H_s(r)}{r}\right) > 1 \quad \rightarrow \quad \text{flaring} \tag{28}$$

$$\frac{d}{dr}\left(\frac{H_s(r)}{r}\right) < 1 \quad \rightarrow \quad \text{self}-\text{shadowed} \tag{29}$$

Some astrophysicists define the case $d(H_s/r)/dr = 0$ as "flat", but I prefer to call that limiting case "conical", because to me flat means $H_s \simeq 0$.

The surface height H_s is related, but not identical to, the pressure scale height H. Typically we have $H_s \simeq 2 \cdots 4H$, and this factor depends on Σ in a rather complicated way which we shall not discuss here.

Self-shadowed disks are only irradiated at their inner edge. The rest of the disk will be practically non-illumated. However, most protoplanetary disks have a flaring geometry, as is already suggested by Eq. (21). So for these disks we can assume that the stellar light can illuminate the surface of the disk, albeit under a very shallow grazing angle. Let us call this angle φ and assume that $\varphi \ll 1$. A typical value is $\varphi \simeq 0.05$.

The flux of stellar radiation at distance r from the star is

$$F_*(r) = \frac{L_*}{4\pi r^2} \tag{30}$$

where L_* is the luminosity of the star. However, the irradiating flux is the *projection* of this flux onto the surface of the disk, which is

$$F_{\rm irr}(r) = \sin(\varphi) F_*(r) \simeq \varphi \frac{L_*}{4\pi r^2} \tag{31}$$

Since we have two sides of the disk we have an irradiation heating of the disk of

$$Q_+^{\rm irr} = \varphi \frac{L_*}{2\pi r^2} \tag{32}$$

Now let us again assume that the disk's effective temperature is exactly enough to thermally radiate this heating away, so let us set $Q_- = Q_+$ with Q_- given by Eq. (25). We then obtain

$$T_{\rm eff} = \left\{ \varphi \frac{L_*}{4\pi \sigma_{\rm SB} r^2} \right\}^{1/4} \tag{33}$$

If we, for convenience, assume that $\varphi(r)$=constant, then we see that

$$T_{\rm eff} \propto L_*^{1/4} r^{-1/2} \tag{34}$$

4.3 Combining the two

The effective temperature due to irradiation drops of shallower than the effective temperature due to accretion. A plot of this is shown in Fig. 1.

If we have both processes acting in concert, then we get

$$Q_- = Q_+^{\rm accr} + Q_+^{\rm irr} \tag{35}$$

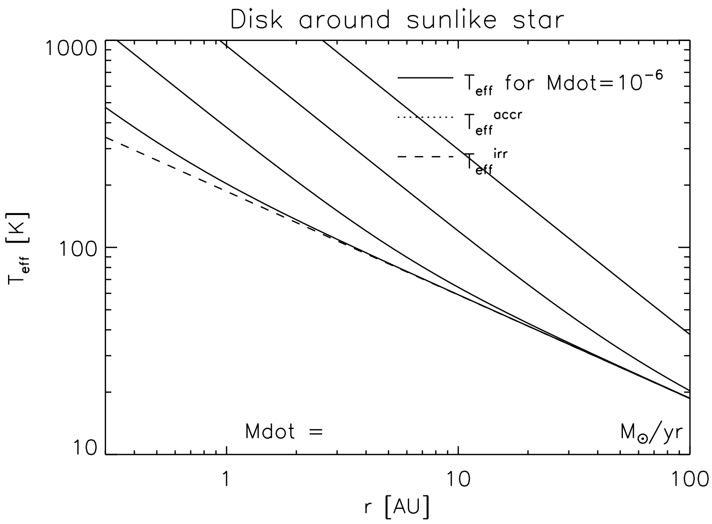

Figure 1. The effective temperature of the disk as a function of radius, for the case where only accretional heating is taken into account (dotted lines) and for the case where only irradiational heating is taken into account (dashed line). Four different accretion rates are shown. The solid line shows the combined $T_{\rm eff}$ from Eq. (36) for $\dot{M} = 10^{-6} M_{\odot}/yr$.

which means for the temperature:

$$T_{\rm eff} = \left\{ \frac{3}{8\pi\sigma_{\rm SB}} \dot{M}\Omega_K^2 + \varphi \frac{L_*}{4\pi\sigma_{\rm SB} r^2} \right\}^{1/4} \tag{36}$$

Since for accretion the $T_{\rm eff}$ drops steeper than for irradiation, we see that $T_{\rm eff}$ is dominated by irradiation for large r, while $T_{\rm eff}$ is dominated by accretion for small r. At which r the turn-over point is depends primarily on L_* and $\dot{M}$.

5 Vertical thermal structure of protoplanetary disks

So far we have only looked at the *effective temperature* of the disk $T_{\rm eff}$. But this does not yet tell the full story, neither for the temperature structure of the disk surface, nor for the interior. So let us now revisit both the accretional heating and the irradiational heating, but this time in a vertical model of the disk. We assume, for simplicity, that the vertical density structure is a Gaussian, like in Section 3, even though we will now deviate from the vertically isothermal assumption. We now need knowledge of the opacity of the dust+gas mixture. Dust opacities are much more important for the thermal structure of the disk than gas opacities (except in the very, very upper layers), so let us focus only on the dust opacities.

5.0.1 Dust opacities

The opacity of dust is strongly frequency dependent. It also depends on the composition. Finally, it depends strongly on the grain size. All these effects are demonstrated in Fig. 2. As you can see, now it becomes ugly. But note that the opacity at stellar wavelengths (optical) tends to be higher than the opacity at infrared wavelengths (perhaps with the exception of the $10\,\mu$m wavelength region).

To simplify our modeling we define κ_* to be the average opacity at stellar wavelength and κ_d to be the average opacity at wavelengths of dust thermal emission.

Since we write the density of the disk usually as the gas density, while the opacities are the dust opacities, we need to use the dust-to-gas ratio. Let us write this as η, and take $\eta = 0.01$. The vertical optical depth of the disk, from the midplane up to infinity, at dust-emission-

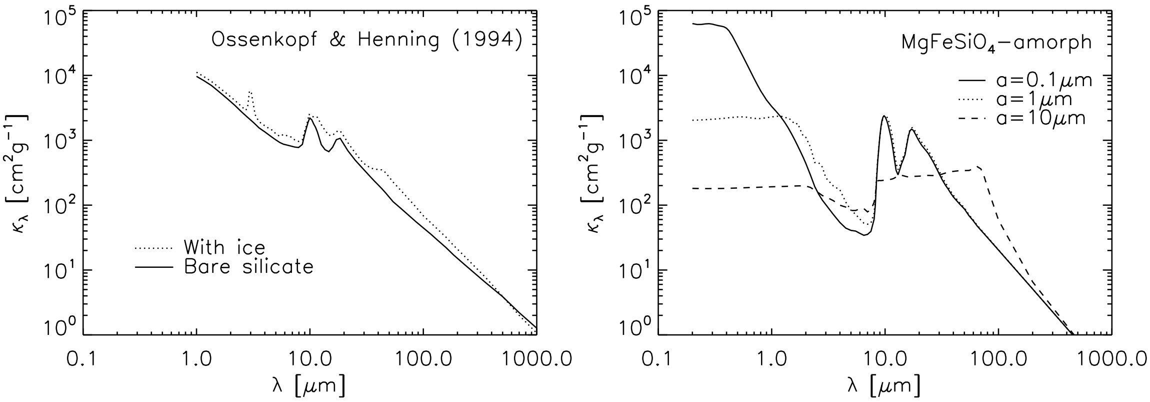

Figure 2. Dust opacities. Left panel: The Ossenkopf & Henning opacities for interstellar medium porous grains, with and without ice coating. Right panel: Olivine grains with 50% magnesium and 50% iron, for different grain sizes.

wavelengths then becomes

$$\tau_d = \frac{1}{2}\Sigma\eta\kappa_d \tag{37}$$

5.0.2 Vertical structure model with disk viscous accretion

Let us first consider the case when there is no irradiation, only viscous heating. At the photosphere of the disk we assume that the temperature is equal to the $T_{\rm eff}$ derived for viscous accretion in Section 4.1. But how does the temperature behave deep down ($z < H_s$)? For that we need radiative diffusion theory, which states that the bolometric radiative flux $F(z)$ is

$$F(z) = -\frac{4\pi}{3\rho\eta\kappa_d}\frac{dJ(z)}{dz} \tag{38}$$

where $J(z)$ is the bolometric mean intensity. Since we are deep in the optically thick regions of the disk, we can assume radiative equilibrium and thus

$$J = \frac{\sigma_{\rm SB}}{\pi}T^4 \tag{39}$$

leading to

$$F(z) = -\frac{4\sigma_{\rm SB}}{3\rho\eta\kappa_d}\frac{dT^4(z)}{dz} \tag{40}$$

Now let us make the simplifying assumption that all the viscously produced heat is released at $z = 0$, and that half of that is diffusing upward, half downward. We focus on the upward half. We then have that $F(z) =$constant and equal to:

$$F = \frac{1}{2}Q_+^{\rm accr} \tag{41}$$

Inserting this into Eq. (40) gives the following differential equation for $T(z)$:

$$\frac{dT^4(z)}{dz} = -\frac{3\rho\eta\kappa_d}{8\sigma_{\rm SB}}Q_+^{\rm accr} \tag{42}$$

This can be directly integrated from $z = H_s$, where we know that the temperature is $T_{\rm eff}$, down to any $z < H_s$. In Figure 3 solutions are shown for a fixed $\dot{M}$, but three values of α (leading to three values of Σ).

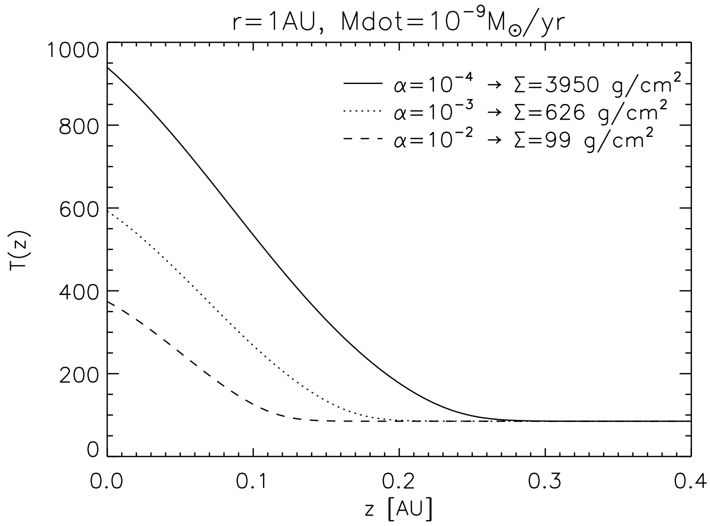

Figure 3. The vertical temperature structure at 1 AU for a protoplanetary disk around a sunlike star for $\dot{M} = 10^{-8} M_\odot$/yr, for $\kappa_d = 1000$ cm^2/g, $\eta = 0.01$, for three different values of α, according to the simple model of Section 5.0.2 where all viscous heat is injected at $z = 0$. You see that high α means (for given $\dot{M}$) a low Σ, hence a low H_s and a low τ, and hence a lower midplane temperature, compared to the cases for lower values of α. If we would have injected the viscous heat not at $z = 0$, but proportional to ρ,t hen the dT/dz at $z = 0$ would be 0.

We can integrate this all the way from $z = H_s$ down to $z = 0$ to obtain the midplane temperature:

$$T_{\rm mid}^4 - T_{\rm eff}^4 = \frac{3\eta\kappa_d}{8\sigma_{\rm SB}} Q_+^{\rm accr} \int_0^{H_s} \rho(z) dz \tag{43}$$

If the photosphere is at $H_s \gg H$, then most of the mass of the disk lies below the photosphere, so that we can say that

$$\int_0^{H_s} \rho(z) dz \simeq \int_0^{+\infty} \rho(z) dz = \frac{1}{2}\Sigma \tag{44}$$

So then we get

$$\begin{aligned} T_{\rm mid}^4 - T_{\rm eff}^4 &= \frac{3\Sigma\eta\kappa_d}{16\sigma_{\rm SB}} Q_+^{\rm accr} \\ &= \frac{3\tau_d}{8\sigma_{\rm SB}} Q_+^{\rm accr} \end{aligned} \tag{45}$$

Since according to Section 4.1 we can say that $T_{\rm eff}^4 = Q_+^{\rm accr}/(2\sigma_{\rm SB})$, which, for $\tau_d \gg 1$, is small compared to the right-hand-side of the above equation, we can write (for $\tau_d \gg 1$) that the midplane temperature is

$$T_{\rm mid} = \left\{ \frac{3\tau_d}{4} \right\}^{1/4} T_{\rm eff} \tag{46}$$

We see that the higher the optical depth is, the higher the midplane temperature for the same accretion rate (i.e. for the same effective temperature for accretion).

5.0.3 Vertical structure model with disk irradiation

Now let us derive what happens when we have a flaring disk with $\varphi = 0.05$. At $z \gg H_s$ any dust grain that happens to be there will see the full radiation of the star $F_* = L_*/(4\pi r^2)$. If the dust grain is very large and $\kappa_d = \kappa_*$, then we can compute the dust temperature using simple geometric arguments. If a is the radius of the (spherical) dust grain, the grain receives $\pi a^2 F_*$

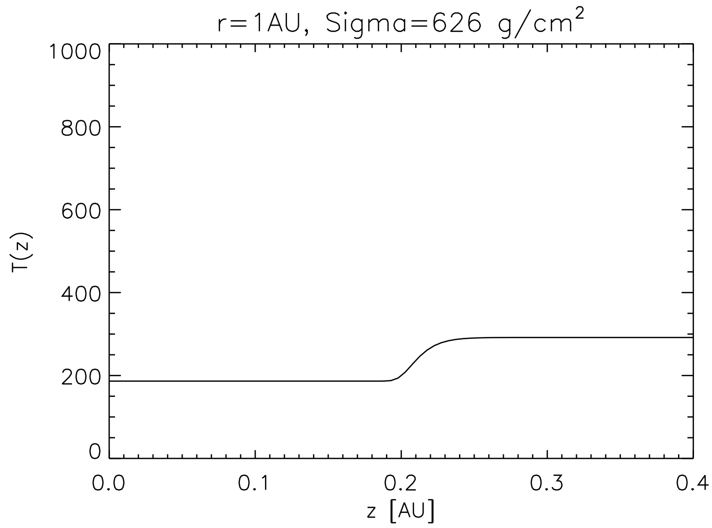

Figure 4. The vertical temperature structure at 1 AU for a protoplanetary disk around a sunlike star for $\Sigma = 626$ g/cm^2, for $\kappa_* = 1000$ cm^2/g, $\eta = 0.01$, according to the simple model of Section 5.0.2. The plot is on the same scale as Fig. 3.

erg/s radiation from the star. If the dust grain has temperature $T_{\rm thin}$, and it can cool by emitting thermal radiation in all directions, it will emit $4\pi a^2\sigma_{\rm SB}T_{\rm thin}^4$ erg/s. Equating the two:

$$\pi a^2 F_* = 4\pi a^2 \sigma_{\rm SB} T_{\rm thin}^4 \tag{47}$$

leads to

$$T_{\rm thin} = \left\{\frac{F_*}{4\sigma_{\rm SB}}\right\}^{1/4} \tag{48}$$

which we call the optically thin dust temperature. If we insert L_* we get

$$T_{\rm thin} = \left\{\frac{L_*}{16\pi r^2\sigma_{\rm SB}}\right\}^{1/4} \tag{49}$$

We can generalize this to smaller grains with $\kappa_d \neq \kappa_*$ in the following way:

$$T_{\rm thin} = \left\{\frac{\kappa_*}{\kappa_d}\frac{L_*}{16\pi\sigma_{\rm SB}}\right\}^{1/4} \tag{50}$$

This also goes as $T_{\rm thin} \propto r^{-1/2}$, like the $T_{\rm eff}$ for the irradiated disk. But we see that typically

$$T_{\rm thin} > T_{\rm eff} \tag{51}$$

This means that dust grains well above the disk are hotter than grains in the disk's photosphere. The surface layers of the disk in fact form a transition from warm dust to cooler dust. They form a *warm surface layer*.

We can make a simple model for this warm surface layer by replacing F_* by $F_*\exp(-\tau_*)$ where

$$\tau_*(z) = \frac{1}{\varphi}\int_z^{infty} \rho(z)\eta\kappa_* dz \tag{52}$$

This means we replace Eq. (50) by

$$T_{\rm surf}(z) = \left\{e^{-\tau_*(z)}\frac{\kappa_*}{\kappa_d}\frac{L_*}{16\pi\sigma_{\rm SB}}\right\}^{1/4} \tag{53}$$

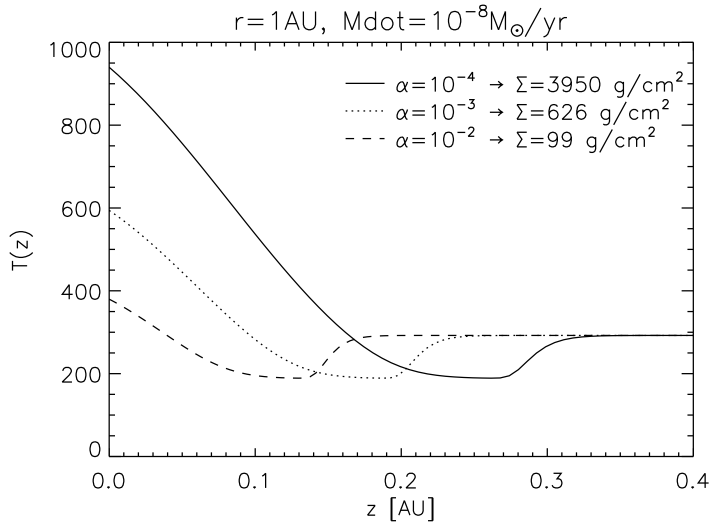

Figure 5. Vertical temperature structures where both viscous heating and irradiation are combined, according to the simple model of Section 5.0.4.

This would go to $T_{\rm surf}(z) \simeq 0$ for $z \to 0$ because $\exp(-\tau_*)$ would become so small. But once we drop below $T^{\rm irr}_{\rm eff}$ we can say that we assume that the temperature is given by $T^{\rm irr}_{\rm eff}$. A smooth transition between these two is given by e.g.:

$$T_{\rm irr}(z) = \left\{T^4_{\rm surf}(z) + T^4_{\rm eff}\right\}^{1/4} \tag{54}$$

This solution is shown in Fig. 4.

5.0.4 Vertical structure model with both viscous heating and irradiation

If we combine the two processes, we can do the same trick as before: we add the temperatures as T^4:

$$T(z) = \left\{T^4_{\rm accr}(z) + T^4_{\rm irr}(z)\right\}^{1/4} \tag{55}$$

The solutions are then shown in Fig. 5.

If we now use these results to plot the midplane temperature as a function of radius, then we find solutions shown in Fig. 6.

6 Location of the snow line

Now we have all the tools necessary to make estimates of the location of the snow line (see Davis 2005). In the early stages of the disk's life time the $\dot{M}$ is still high, and according to Fig. 6 this means that the snow line must lie at very large radii. However, as $\dot{M}$ drops, the snow line moves inward. To really calculate the snow line we have to also account for the gas pressure in the disk. The higher the pressure, the higher the temperature at which ices can survive.

It is also important to realize that even if ice cannot exist at the midplane and in the very surface, there might be an intermediate z where the ice can survice. This leads to a kind of "cloud deck" in the disk, as shown in the models by Davis and by Min et al. In Fig. 7 I reproduce one of the figures of Min et al.

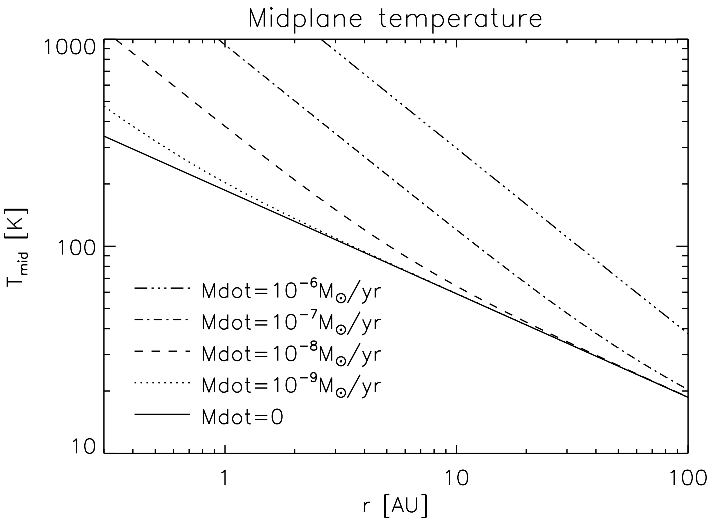

Figure 6. The midplane temperature of the disk as a function of radius, according to the simple model of Section 5.0.4.

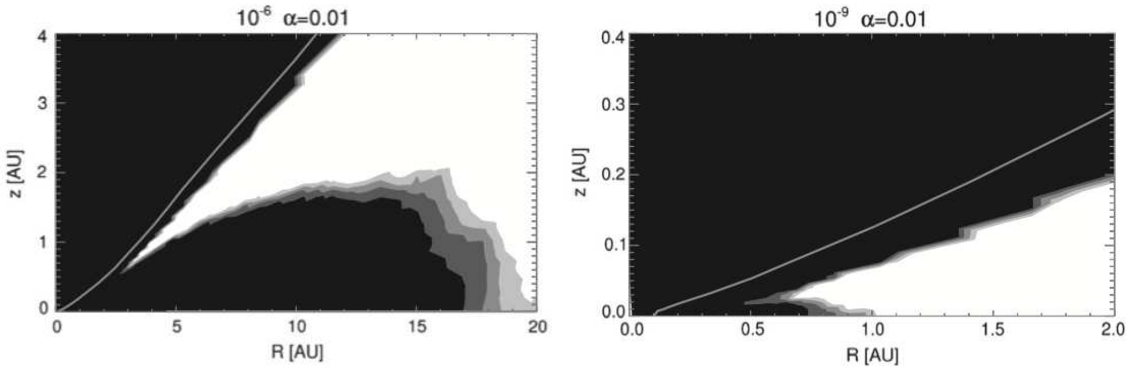

Figure 7. Figures from Min, Dullemond, Kama & Dominik (2011), Icarus 212, 416. Shown in white is the region where water ice can survive in a disk with irradiation and viscous heating by accretion. You see that for $\dot{M} = 10^{-6} M_{\odot}$/yr a "cloud deck" is produced and the ice line lies very far out. For $\dot{M} = 10^{-9} M_{\odot}$/yr no such "cloud deck" is seen and the ice line is very near the Earth-forming region.

A Revisiting the α-viscosity prescription

In Section 2 we introduced the α-viscosity prescription in a rather ad-hoc way. Here we redo this a bit better. The main idea is that in a normal ideal gas viscosity is brought about by molecules moving some mean free path $\lambda_{\rm free}$ before they collide with another molecule and exchange momentum with that molecule. This allows momentum to be transported over small, but non-negligible, distances. The viscosity is then proportional to:

$$\nu_{\rm mol} = A\lambda_{\rm free} v_{\rm therm} \tag{56}$$

where $v_{\rm therm}$ is the thermal velocity of the gas particles and $A \simeq 1$ is some dimensionless proportionality constant. The dimensions work out: $[\nu]$ =cm^2/s, while $[\lambda_{\rm free}]$ =cm and $[v_{\rm therm}]$ =cm/s.

In a turbulent flow we can say that turbulent eddies also have a certain "mean free path" L. This is the distance they travel before they dissipate. This is typically *much* larger than the molecular mean free path, i.e. $L \gg \lambda_{\rm free}$. The speed V at which the turbulent eddies move is, however, typically a bit slower than that of the molecules, but not by much. With this picture in mind we can say that the turbulence acts as a kind of large-scale "viscosity" with

$$\nu_{\rm turb} = LV \tag{57}$$

Let us now say that

$$L = \beta H \qquad \text{and} \qquad V = \gamma c_s \tag{58}$$

where H is the pressure scale height of the disk ($H = c_s/\Omega_K$). This gives

$$\nu = \beta\gamma H c_s = \beta\gamma \frac{c_s^2}{\Omega_K} \tag{59}$$

For consistency reasons we must have $\beta < 1$ (it is hard to imagine a turbulent eddy larger than the pressure scale height) and $\gamma < 1$ (it is hard to imagine a turbulent eddy that does supersonic). Since it is evidently only important to know the product of β and γ, it is customary to write

$$\nu = \alpha H c_s = \alpha \frac{c_s^2}{\Omega_K} \tag{60}$$

instead.

B References

The theory given in this lecture is assembled from many building blocks from the literature, among which are the following:

- Shakura & Sunyaev (1973) A&A 24, 337 on basic theory of viscous disk accretion and the α-viscosity prescription
- Lynden-Bell & Pringle (1974) MNRAS 168, 603 on a simple self-similar accretion-spreading model of disk evolution
- Chiang & Goldreich (1997) ApJ 490, 368 on a simple but powerful model for an irradiated disk
- D'Alessio et al. (1998) ApJ 500, 411 on detailed 1+1D disk structure models
- Hartmann et al. (1998) ApJ 495, 385 on comparing evolving disk models to observations of disks
- Davis (2005) ApJ 620, 994 on the location of the snow line, and how that location evolves with time

- The book “Accretion power in astrophysics” by Frank, King & Raine.

Note that much of the theory is by now so standardized that it might well be that I have overlooked/forgotten an important reference. Please let me know if I did.

Recommended further literature:

- Hueso & Guillot (2005) A&A 442, 703 on a model of star formation with disk formation and spreading

Some examples of more recent models (highly biased list):

- Min et al. (2011) Icarus 212, 416 on 2-D/3-D radiative transfer modeling of protoplanetary disks with accretion and irradiation, focusing on the location of the snow line.
- Siebenmorgen et al. (2012) A&A on shadows, gaps and ring-like structures in proto-planetary disks
- Pinte et al. (2009) A&A 498, 967 on a benchmark model for continuum radiative transfer in highly optically thick protoplanetary disks